SECTION 10.8: CHOOSING TESTS

Suppose you're given a series: $\sum_{k=N}^{\infty} a_k$

- **THE DIVERGENCE TEST:** If $\lim_{k\to\infty} a_k \neq 0$, then the series **diverges**. If $\lim_{k\to\infty} a_k = 0$, **WE KNOW NOTHING!**

- **GEOMETRIC SERIES:** If the terms of the series involve exponential functions, see if it is a geometric series.

  Find the common ratio, $r = \dfrac{a_{k+1}}{a_k}$:

  - if $|r| < 1$, the series **converges absolutely**
  - if $|r| \geq 1$, the series **diverges**.

- ***p*-series:** Is the series a p-series, $\sum_{k=N}^{\infty} \dfrac{c}{k^p}$? If so, it **converges** if $p > 1$ and **diverges** if $p \leq 1$.

- **LIMIT COMPARISON TEST (LCT):** If the series is 'close' to being a p-series, find the p-series obtained by computing the ratio of the leading term in the numerator divided by the leading term in the denominator. Use the LCT with this p-series and your original series. If the limit of the ratio of these series is **finite** and **positive**, the two series either **both converge** or **both diverge**.

- **ALTERNATING SERIES TEST (AST):** If $\sum_{k=N}^{\infty} a_k$ is alternating and $|a_k|$ decrease to 0, then the series converges.

- **RATIO TEST:** If the series involves factorials, use the ratio test.

  Find: $\lim_{k\to\infty} \left| \dfrac{a_{k+1}}{a_k} \right| = L$:

  - if $L < 1$, it **converges absolutely**
  - if $L > 1$ or $L = \infty$, it **diverges**

  NOTE: If $L = 1$, **WE KNOW NOTHING!**

- **ROOT TEST:** If the series involves variables to variable powers, try the root test.

  Find $\lim_{k\to\infty} \sqrt[k]{|a_k|} = L$:

  - if $L < 1$, it **converges absolutely**
  - if $L > 1$ or $L = \infty$, it **diverges**

  NOTE: If $L = 1$, **WE KNOW NOTHING!**

- **OTHER TESTS:** Direct Comparison Test (DCT), Extreme Cases of the Limit Comparison Tests, The Integral Test.

Always remember: 'When in doubt, write it out!'

EXAMPLE: Determine if the given series is absolutely convergent (AC), conditionally convergent(CC), or divergent (D).

In each case, state which test you used and a detailed explanation.

1. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k\sqrt[4]{k}}$

2. $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^{k+1}2^k}{3^{k-1}}$

3. $\displaystyle\sum_{k=2}^{\infty} \frac{k}{\sqrt[3]{k^7+1}}$

4. $\displaystyle\sum_{k=1}^{\infty} \frac{(-4)^k}{k!}$

5. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{1+e^{-k}}$

6. $\displaystyle\sum_{k=15}^{\infty} \frac{(-1)^{k+1}}{2k-1}$

7. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{2k}{3k+1}\right)^k$

8. $\displaystyle\sum_{k=1}^{\infty} \frac{\cos(2k)}{k^{1.001}}$

9. $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k\ln k}$

SOLUTIONS:

1. AC: p-series with $p = 5/4 > 1$.

2. AC: Geometric Series with $|r| = |-2/3| = 2/3 < 1$.

3. AC: Compare with $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^{4/3}}$ - a convergent p-series with $p = 4/3 > 1$:
$$\lim_{k\to\infty} \frac{\frac{k}{\sqrt[3]{k^7+1}}}{\frac{1}{k^{4/3}}} = \lim_{k\to\infty} \frac{k^{7/3}}{\sqrt[3]{k^7+1}} = \lim_{k\to\infty} \frac{k^{7/3}}{k^{7/3}\sqrt[3]{1+1/k^{7/3}}} = 1 > 0,$$
so the series converges by the L.C.T.

4. AC: $\displaystyle\lim_{k\to\infty} \frac{\left|\frac{(-4)^{k+1}}{(k+1)!}\right|}{\left|\frac{(-4)^k}{k!}\right|} = \lim_{k\to\infty} \frac{4}{k+1} = 0 < 1$, so the series converges absolutely by the Ratio Test.

5. D: $\displaystyle\lim_{k\to\infty} \frac{1}{1+e^{-k}} = 1 \neq 0$, so the series diverges by the divergence test.

6. CC: For $\displaystyle\sum_{k=15}^{\infty} \left|\frac{(-1)^{k+1}}{2k-1}\right| = \sum_{k=15}^{\infty} \frac{1}{2k-1}$, compare with $\displaystyle\sum_{k=15}^{\infty} \frac{1}{k}$, a divergent p-series with $p = 1 \leq 1$:

   $\displaystyle\lim_{k\to\infty} \frac{\frac{1}{2k-1}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k}{2k-1} = \frac{1}{2} > 0$, so the series $\displaystyle\sum_{k=15}^{\infty} \frac{1}{2k-1}$ diverges using the L.C.T.

   So $\displaystyle\sum_{k=15}^{\infty} \frac{(-1)^{k+1}}{2k-1}$ is not AC. However, $\displaystyle\sum_{k=15}^{\infty} \frac{(-1)^{k+1}}{2k-1}$ is alternating with $|a_k| = \dfrac{1}{2k-1}$, $k \geq 15$:

   (a) $|a_k|$ are decreasing since $f(x) = \dfrac{1}{2x-1}$ has $f'(x) = -\dfrac{2}{(2x-1)^2} < 0$ for $x \geq 15$, and

   (b) $\displaystyle\lim_{k\to\infty} |a_k| = \lim_{k\to\infty} \frac{1}{2k-1} = 0$,

   so $\displaystyle\sum_{k=15}^{\infty} \frac{(-1)^{k+1}}{2k-1}$ converges by the A.S.T. Hence $\displaystyle\sum_{k=15}^{\infty} \frac{(-1)^{k+1}}{2k-1}$ is conditionally convergent

7. AC: $\displaystyle\lim_{k\to\infty} \sqrt[k]{\left|\frac{2k}{3k+1}\right|^k} = \lim_{k\to\infty} \frac{2k}{3k+1} = \frac{2}{3} < 1$, so the series converges absolutely by the Root Test.

8. AC: Compare with $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{1.001}}$, a convergent p-series with $p = 1.001 > 1$: $0 \leq \left|\dfrac{\cos(2k)}{k^{1.001}}\right| \leq \dfrac{1}{k^{1.001}}$ for all $k \geq 1$, so the series converges absolutely by the D.C.T.

9. D: Let $f(x) = \dfrac{1}{x \ln x}$. Then:

   (a) f is positive for $x \geq 3$.

   (b) $f'(x) = \dfrac{-1-\ln x}{(x\ln x)^2} < 0$ for all $x \geq 3$.

   (c) f is continuous (since it's differentiable!) for $x \geq 3$.

   $\displaystyle\int_3^{\infty} \frac{1}{x\ln x}\,dx = \lim_{b\to\infty} \int_3^b \frac{1}{x\ln x}\,dx = \lim_{b\to\infty} (\ln|\ln b| - \ln|\ln 3|) = \infty.$

   Hence, the series diverges by the integral test.